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A GALERKIN PROCEDURE FOR LOCALIZED BUCKLING OF A STRUT ON A NONLINEAR ELASTIC FOUNDATION

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Abstract—Working from the governing differential equation a Galerkin procedure has been developed to model the localized buckle pattern of an elastic strut on a nonlinear elastic (softening) Winkler foundation. The choice of trial functions comes directly from a perturbation study about the critical point, the actual shape being determined by the eigenvalues of the linearized equation. A good comparison has been achieved with independently obtained numerical solutions using only a few modes. Other measures of the accuracy of the approximate solutions have also proven to be favourable. Copyright © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Localized buckling occurs in long elastic structures which have an equal propensity to buckle at any location along their length. Examples of such behaviour are the thermally-induced buckling of submarine pipelines and railway tracks (Tvergaard and Needleman (1981)). Although initially triggered by a material or geometric imperfection such behaviour is an intrinsic property of the perfect system.

The examples above may be modelled as long elastic beam-columns resting on elastic foundations and subjected to axial compression (Hetényi (1946); Bažant and Cedolin (1991)). Under conservative loading this is a problem of elastic stability which has a hamiltonian nature and is known to exhibit a great diversity of solutions (Hunt and Wadee, (1991); Champneys and Toland (1993)). Despite the existence of a multiplicity of post-critical equilibrium states to the governing differential equation (Fig. 1), the actual buckled shape adopted by a structure is governed by the principle of least energy. In general, for long structures this corresponds to the single-hump localized response which is the subject of this work. Important early contributions to the study of localized buckling of flexible beams on nonlinear elastic foundations have been made by many investigators (Tvergaard and Needleman (1980); Potier-Ferry (1983)).

Traditionally, engineers seeking to understand localized buckling of struts on foundations have used perturbation methods to study the behaviour close to the critical point where the fundamental equilibrium path intersects the unstable symmetric post-buckling path (Thompson and Hunt (1973)). Such schemes, which begin with either the differential equation or an energy formulation, yield solutions in terms of a perturbation parameter, which is a measure of the proximity to the exact linear eigenvalue solution at the critical point, and are therefore valid only in an asymptotic sense (Murdock (1991)).

This paper demonstrates a technique which obviates this dependence upon the critical point and is capable of generating good approximate solutions throughout the post-buckling régime. The localized buckle of an infinitely long elastic strut is modelled using a Galerkin approach in which the deflected shape is represented by a sum of continuous displacement functions. The general form of these functions is suggested by a perturbation study, while the exact shape is determined by the real and imaginary parts of the linearized eigenvalues at the appropriate load value. The amplitudes of each mode form, representing the generalized displacements of the deformed structure, are determined using Galerkin's method.

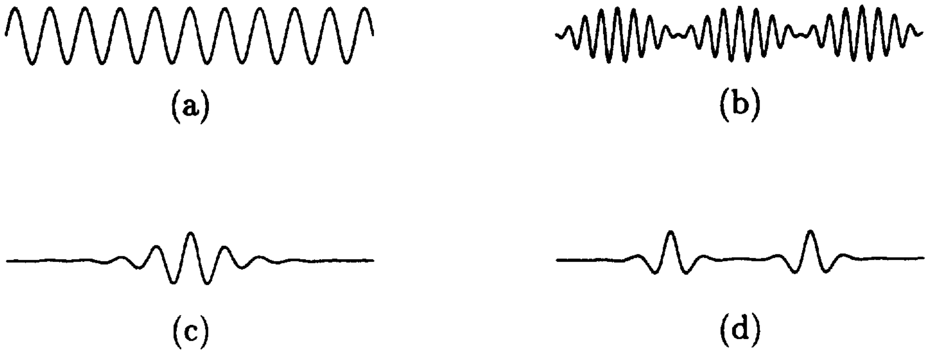

Fig. 1. A variety of solutions exist for the nonlinear differential equation $EIw'''' + Pw'' + kw - cw^3 = 0$ including: (a) periodic; (b) modulated; (c) single-hump localized; and (d) multi-hump localized responses.

The Galerkin procedure is one of many classical methods of analysis originally developed for the manual solution of problems with simple geometries. With the advent of cheap computers have come more general numerical methods capable of solving more complicated geometries. In this case, however, there are several advantages in using a classical modal approach. The one-dimensional strut model results in the simplest possible geometry, and *a priori* knowledge, gained from previous perturbation studies, may be incorporated in the modal form. Related problems with wavelength-dependent foundations (Hunt *et al.* (1995)) may be tackled more directly using a modal approach. Near to the critical point, where the approximate solutions are most accurate, initial value- and boundary value solvers become difficult to use as the truncation length increases beyond practical bounds (Champneys and Spence (1993)). In any case, differential equation solvers require a reasonably accurate initial guess to converge to the solution of nonlinear problems. An approximate solution, using only a few modes, can be used as a starting point for a numerical scheme, and, with additional modes, may provide an independent check of the numerical solution.

This work is the first step in a project to investigate localized buckling of elastic structures in visco-elastic media. The presence of a velocity-dependent constitutive relation results in the action of non-conservative forces so that a potential energy formulation is not possible. For this reason the Galerkin approach, utilizing the differential equation, has been used here in favour of other approximate procedures such as the Rayleigh–Ritz method.

A scenario currently being explored involves a strut, subjected to a compressive load, resting on a Maxwell foundation (represented by a spring and dashpot in series). This model exhibits an instantaneous elastic buckle followed by a period of visco-elastic behaviour, and can be used to describe the formation of folding in geological strata. A first foray into this phenomenon has recently been published (Hunt *et al.* (1995)).

2. FORMULATION OF THE DIFFERENTIAL EQUATION

The differential equation governing the response of a compressed strut (Fig. 2) may be derived either: directly by equilibration of forces; or from the Principle of Virtual Work; or by using an energy formulation. The latter approach is adopted here with the total potential energy for a linearized strut on a symmetric softening (cubic) foundation being

$$V = \int_{-\infty}^{\infty} \mathscr{F}(w, w', w'')\, dx, \tag{1}$$

where a prime (′) is used to represent differentiation with respect to the spatial variable, x. The integrand is

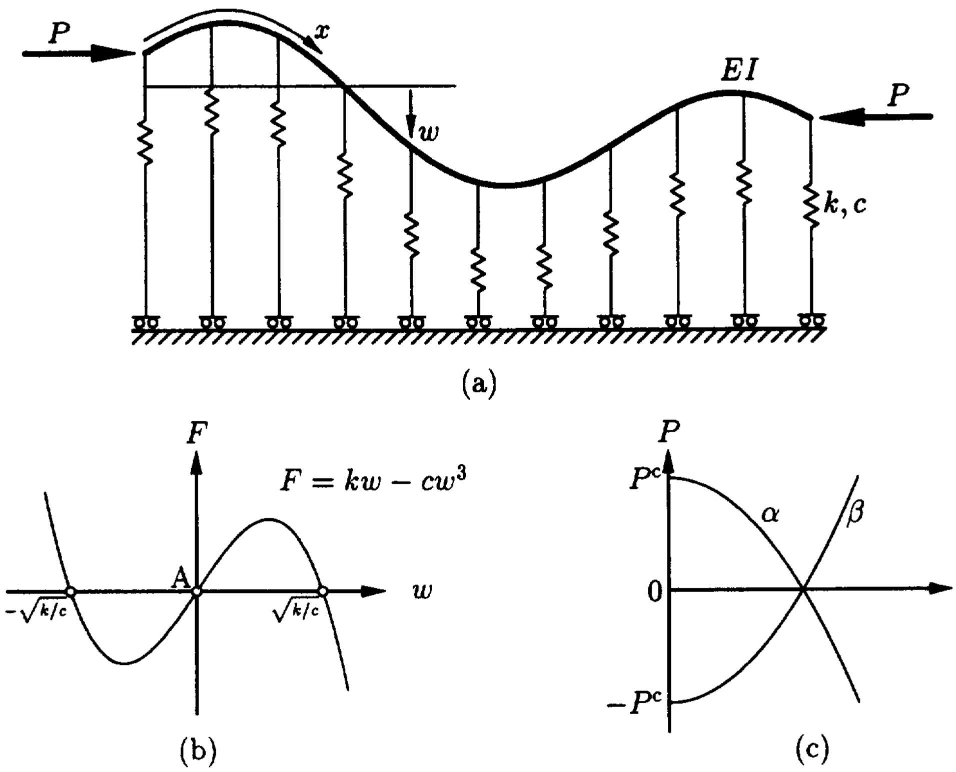

Fig. 2. (a) An elastic strut resting on a nonlinear (softening) elastic Winkler foundation; (b) symmetric foundation resistance; (c) the root structure of the fundamental equilibrium state A, where α and β define respectively the real and imaginary parts of the eigenvalue.

$$\mathscr{F} = \tfrac{1}{2}EIw''^2 - \tfrac{1}{2}Pw'^2 + \tfrac{1}{2}kw^2 - \tfrac{1}{4}cw^4, \tag{2}$$

in which EI is the bending stiffness of the strut, P is the axial compression, and k and c are the linear and nonlinear components of the foundation stiffness respectively. Higher-order effects in both the bending energy and work done by load have been omitted (Thompson and Hunt (1973)). Application of the calculus of variations to the total potential energy (1) results in the dimensional form of the differential equation

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + kw - cw^3 = 0. \tag{3}$$

2.1. *Nondimensionalization*

It is useful to nondimensionalize the differential equation so that solutions are independent of the relative magnitudes of material parameters. By using the following nondimensional parameters

$$\tilde{P} = \frac{P}{P^c}, \quad \tilde{w} = \sqrt{\frac{c}{k}}w, \quad \tilde{x} = \omega^c x, \tag{4}$$

where the superscripts are used to denote those quantities evaluated at the critical point,

$$P^c = 2\sqrt{kEI}, \quad w^c = \sqrt[4]{\frac{k}{EI}}, \tag{5}$$

the resulting nondimensional form is obtained

$$\frac{d^4\tilde{w}}{d\tilde{x}^4}+2\tilde{P}\frac{d^2\tilde{w}}{d\tilde{x}^2}+\tilde{w}-\tilde{w}^3 = 0. \tag{6}$$

From now on the tildes above each term shall be dropped and the dimensionless form of the differential equation,

$$w''''+2Pw''+w-w^3 = 0, \tag{7}$$

will be used, in which $()' \equiv d/dx$. Whilst a numerical solution of this differential equation may be obtained readily using standard numerical integration schemes, it appears almost certain that a closed-form solution does not exist. The task is therefore to develop an approximate procedure which is much simpler to apply than the numerical techniques, and yet which retains much of the underlying character of the solution.

2.2. *Linearized conditions*

When w is small, as in the tails of the localized solution ($x \to \pm\infty$), the linearized form of eqn (7) must be satisfied. Adopting a general complex representation $w(x) = A\,e^{\lambda x}$, where $\lambda = \alpha + i\beta$, results in the characteristic equation

$$\lambda^4+2P\lambda^2+1 = 0. \tag{8}$$

After substituting for λ, and equating the real and imaginary parts to zero, the eigenvalues of the linearized system can be expressed as

$$\alpha = \sqrt{\frac{1-P}{2}} \quad \beta = \sqrt{\frac{1+P}{2}}, \tag{9}$$

where α and β define, respectively, the rate of exponential growth and decay, and the wavelength of the modulated periodicity at the tails of the localization. It is these quantities which are used to describe the precise shape of the mode form suggested by the perturbation method of the following section.

3. PERTURBATION ANALYSIS

A nonlinear double-scale perturbation analysis of the differential eqn (7) can be used to determine the load and form of the solution at the critical point, and also for developing an ordered sequence of linear differential equations to describe the post-buckled response (Wadee (1993)). The scheme, outlined only briefly here, is derived from the total potential energy (1) and reduces the problem to a power series in terms of a perturbation parameter, s, which is a measure of the distance away from the critical point. The beauty of using an energy formulation as the basis for the perturbation method is that the suppression of secular terms and elimination of passive co-ordinates are guaranteed automatically. A detailed account of the method is to be published in a forthcoming paper (Wadee *et al.* (1996)).

The deflected shape is expanded as a Fourier series in x

$$w = \sum_{i=0}^{\infty} \{A_i(X)\cos i\omega x + B_i(X)\sin i\omega x\}, \tag{10}$$

where the amplitudes, A_i and B_i, are permitted to vary slowly by expanding as power series in the perturbation parameter to give

$$A_i = A_{i,1}(X)s + A_{i,2}(X)s^2 + \cdots,$$
$$B_i = B_{i,1}(X)s + B_{i,2}(X)s^2 + \cdots. \tag{11}$$

Here, $X = sx$ is the so-called slow-space variable, and is a function of both x and the perturbation parameter, s, which is defined (Hunt *et al.* (1989)) as

$$s = \sqrt{P^c - P}. \tag{12}$$

The load and frequency, P and ω, are also expanded as power series in s,

$$P = P^c + P_1 s + P_2 s^2 + \cdots,$$
$$\omega = \omega^c + \omega_1 s + \omega_2 s^2 + \cdots. \tag{13}$$

After substituting the expanded form of the deflected shape (10) into the integrand (2), and applying the calculus of variations, equations are obtained at successive powers of s for the unknown amplitudes, A_i and B_i. In principle, the scheme can be continued to an arbitrary level of s, although obtaining a closed-form solution is likely to become increasingly more difficult.

To enable closed-form solutions to be found, so-called *contour integrals*, having the form of the second and third integrals in eqn (25), are ignored. While this simplification leads to negligible error adjacent to the critical point, there is a growing contribution from these integrals further into the post-buckling régime. These integrals are included in the subsequent Galerkin analysis where they are evaluated in the complex plane by applying the calculus of residues.

The expression obtained for the deflection to order s^4 is

$$\begin{aligned} w = {} & sA_{1,1}\cos\omega x \\ & + s^2 B_{1,2}\sin\omega x \\ & + s^3(A_{1,3}\cos\omega x + A_{3,3}\cos 3\omega x) \\ & + s^4(B_{1,4}\sin\omega x + B_{3,4}\sin 3\omega x) \\ & + O(s^5), \end{aligned} \tag{14}$$

where the amplitudes are

$$A_{1,1} = \frac{4\omega^c}{\sqrt{6c}}\operatorname{sech}\Omega X,$$

$$B_{1,2} = \frac{\omega^c}{\sqrt{P^c}}\sqrt{\frac{3}{c}}\operatorname{sech}\Omega X\tanh\Omega X,$$

$$A_{1,3} = \frac{1}{72\sqrt{6c}}\frac{1}{\omega^c EI}(-317\operatorname{sech}\Omega X + 307\operatorname{sech}^3\Omega X),$$

$$A_{3,3} = \frac{\omega^{c3}}{24k\sqrt{6c}}\operatorname{sech}^3\Omega X,$$

$$B_{1,4} = \frac{1}{864\sqrt{6c}}\frac{1}{EI\sqrt{k}}(-2389\operatorname{sech}\Omega X\tanh\Omega X + 5524\operatorname{sech}^3\Omega X\tanh\Omega X),$$

$$B_{3,4} = \frac{\omega^{c3}}{16k\sqrt{P^c}}\sqrt{\frac{3}{c}}\operatorname{sech}^3 \Omega X \tanh \Omega X. \tag{15}$$

In the expressions above

$$\Omega = \frac{\omega^c}{\sqrt{2P^c}}. \tag{16}$$

In general, perturbation results may be extended to any desired level of accuracy, however, in this instance their applicability is limited because contour integrals have been ignored.

4. THE GALERKIN PROCEDURE

The Galerkin procedure, which belongs to the family of techniques known collectively as the method of weighted residuals, can reveal many of the qualitative features of a solution using only a few modes. More precise answers can be achieved by introducing additional modes (Finlayson (1972)). An approximate solution, $\bar{w}$, consisting of a number of modes ϕ_i (also known as trial functions or displacement shapes), is chosen to represent the deflected shape of the strut

$$\bar{w}(x) = \sum_{i=0}^{n} A_i \phi_i, \tag{17}$$

with the centre of the localization (sometimes called the *symmetric section*) assumed to occur at $x = 0$. Owing to the approximate nature of the deflected shape, the differential eqn (7) will not be satisfied exactly. The remainder, which is known as the *residual*, is

$$R(x, \bar{w}) = \bar{w}'''' + 2P\bar{w}'' + \bar{w} - \bar{w}^3, \tag{18}$$

and depends not only on the form of the approximate solution but varies also with the position along the length of the strut. The quantity

$$\int_{-\infty}^{\infty} R(x, \bar{w})\phi_i \, \mathrm{d}x \tag{19}$$

has the dimensions of work and is defined as the excess energy of the strut over the displacement ϕ_i. The Galerkin approximation requires this virtual excess energy to vanish which leads to simultaneous nonlinear algebraic equations in terms of the unknown amplitudes, A_i.

4.1. *Choice of mode shapes*

The modes, ϕ_i, are chosen to satisfy the flat boundary conditions expected of a localized solution at $x = \pm\infty$, although this is not strictly necessary for the Galerkin procedure. While there are many such functions, the greatest accuracy, for a small number of modes, is obtained by using experience to select the most suitable form. Greater accuracy can also be attained by the increasing the number of modes, though naturally at the expense of a corresponding increase in the number of equations to be solved.

The perturbation method outlined previously provides approximate closed-form solutions which are valid close to the critical point. In the procedure that follows, the form of these perturbation solutions is maintained throughout the post-buckled state with the actual shape at different load levels being governed by the eigenvalues of the linearized differential equation. The effect of the contour integrals, defined previously, is included, thus ensuring the validity of the analysis throughout the post-buckled state, unlike the results of the earlier perturbation method. The amplitudes A_i of eqn (17), which are constant for fixed values of P, should not be confused with the slowly varying amplitudes, A_i and B_i, of the perturbation approach in eqn (11), which depend on P and x, despite the similar notation.

According to Galerkin's method, trial functions must be linearly independent and as $n \to \infty$ the approximate solution should tend to the exact solution. Whilst the first criterion is a natural consequence of the combined trigonometric and hyperbolic mode shapes chosen, a formal proof of the second is not performed—the results themselves being considered sufficient justification.

4.2. *First mode*

The first term in the ordered perturbation analysis (14) is used as the single mode approximation so that

$$\phi_1 = \operatorname{sech} \alpha x \cos \beta x. \tag{20}$$

The approximate displacement according to eqn (17) is then

$$\bar{w} = A_1 \phi_1, \tag{21}$$

where the amplitude, A_1, is found from the Galerkin equation

$$\int_{-\infty}^{\infty} R(x, \bar{w}) \phi_1 \, dx = 0. \tag{22}$$

This equation indicates that the work done by the residual (out of balance) forces over the displacements $\phi_1(x)$ is zero.

For a particular value of load, P, the precise shape of the mode (20) is determined by the α and β of the linearized system given in eqn (9), and after substituting for $\bar{w}$ and integrating over the infinite domain, the result is a nonlinear algebraic equation in terms of the unknown amplitude, A_1. The Galerkin equation is

$$\begin{aligned} &\tfrac{1}{2} f_1 A_1 I_2 + \tfrac{1}{2} f_1 A_1 I_{c,22} + \tfrac{1}{2} f_2 A_1 I_{s,22} + \{2\alpha^2(-5\alpha^2 + 3\beta^2 - P) A_1 - \tfrac{3}{8} A_1^3\} I_4 \\ &\quad + \{2\alpha^2(-5\alpha^2 + 3\beta^2 - P) A_1 - \tfrac{1}{2} A_1^3\} I_{c,42} - 12\alpha^3 \beta A_1 I_{s,42} - \tfrac{1}{8} A_1^3 I_{c,44} \\ &\quad + 12\alpha^4 A_1 I_6 + 12\alpha^4 A_1 I_{c,62} = 0 \end{aligned} \tag{23}$$

where f_1 and f_2 are

$$\begin{aligned} f_1 &= \alpha^4 - 6\alpha^2\beta^2 + \beta^4 + 2P(\alpha^2 - \beta^2) + 1 = 0, \\ f_2 &= 4\alpha\beta(\alpha^2 - \beta^2 + P) = 0, \end{aligned} \tag{24}$$

and the various integrals are represented by

$$\begin{aligned} I_n &= \int_{-\infty}^{\infty} \operatorname{sech}^n \alpha x \, dx, \\ I_{c,nm} &= \int_{-\infty}^{\infty} \operatorname{sech}^n \alpha x \cos m\beta x \, dx, \\ I_{s,nm} &= \int_{-\infty}^{\infty} \operatorname{sech}^n \alpha x \tanh \alpha x \sin m\beta x \, dx. \end{aligned} \tag{25}$$

These integrals may be evaluated directly, or derived from, standard integrals published in various texts (Gradshteyn and Ryzhik (1994)). Alternatively the calculus of residues (Stephenson and Radmore (1990)) is a very useful tool for evaluating integrals over an infinite domain.

Rearranging eqn (23) in terms of the amplitude A_1 and substituting for the integrals above gives

$$k_1 A_1 - k_{111} A_1^3 = 0, \tag{26}$$

where

$$\begin{aligned} k_1 &= \frac{8\alpha}{15}(-\alpha^2 + 15\beta^2 - 5P) \\ &\quad + \frac{8\beta\pi}{15\alpha^2}(\alpha^2+\beta^2)(-\alpha^2+6\beta^2-5P)C_1, \\ k_{111} &= \frac{1}{2\alpha} + \frac{2\beta\pi}{3\alpha^4}(\alpha^2+\beta^2)C_1 + \frac{\beta\pi}{3\alpha^4}(\alpha^2+4\beta^2)C_2, \end{aligned} \tag{27}$$

in which the following substitutions have been made

$$C_1 = \operatorname{cosech}\frac{\beta\pi}{\alpha}, \quad C_2 = \operatorname{cosech}\frac{2\beta\pi}{\alpha}. \tag{28}$$

The amplitude A_1 is then found directly as

$$A_1 = \sqrt{\frac{k_1}{k_{111}}}. \tag{29}$$

A closed-form solution for the amplitude is a unique property of the single mode approximation as coupled equations result when the deflected shape is represented by two or more mode shapes.

4.3. *Second mode*

A more accurate solution can be obtained by incorporating two modes in the approximate deflected shape. In addition to mode ϕ_1, a second mode,

$$\phi_2 = \operatorname{sech}\alpha x \tanh \alpha x \sin\beta x, \tag{30}$$

corresponding to the s^2 coefficient in the perturbation analysis (14), is introduced. This term allows a modulation in phase which is especially important around the symmetric section (Wadee (1993)) where the affect of the nonlinearity is greatest. Substitution of the second mode into eqn (17), to determine the corresponding Galerkin equations, requires much algebraic manipulation which can be performed most simply using a computation package such as Mathematica (Wolfram (1988)). After substituting for the integrals (25) the amplitudes A_i can be determined from the coupled equations

$$\begin{aligned} k_1A_1 + k_2A_2 + A_1(k_{111}A_1^2 + k_{122}A_2^2) + A_2(k_{211}A_1^2 + k_{222}A_2^2) &= 0, \\ l_1A_1 + l_2A_2 + A_1(l_{111}A_1^2 + l_{122}A_2^2) + A_2(l_{211}A_1^2 + l_{222}A_2^2) &= 0, \end{aligned} \tag{31}$$

where the coefficients of the amplitudes are listed in the Appendix. These expressions have been programmed in the computer language "C" and solved iteratively using a Newton–Raphson algorithm (Press *et al.* (1988)). The resulting solutions are expected to provide a better estimate of the numerical solution than those arising from the single mode approximation.

4.4. *Higher modes*

Once again the perturbation analysis (14) is used as a guide for selecting higher order mode shapes. Passive terms, such as $\cos 3\omega x$ and $\sin 3\omega x$ in eqn (14), arising from the cubic form of the nonlinearity in the foundation stiffness are ignored owing to their meagre contribution to the deflected shape of the strut. This is justified by comparing the relative magnitudes of the coefficients of the $\text{sech}^3 \Omega X$ terms in the perturbation amplitudes of eqn (15). The magnitude of the active mode, $A_{1,3}$, is approximately 100 times greater than for the corresponding passive mode, $A_{3,3}$.

The third and fourth modes are therefore taken to be

$$\phi_3 = \text{sech}^3\, \alpha x \cos \beta x,$$

$$\phi_4 = \text{sech}^3\, \alpha x \tanh \alpha x \sin \beta x, \tag{32}$$

where ϕ_3 allows a correction to the amplitude and ϕ_4 an associated phase modulation. The corresponding Galerkin equations are not presented here on account of their length but the results are described in the following section.

5. DISCUSSION OF RESULTS

It is not possible to obtain the *exact* solution to the nonlinear differential eqn (7). Instead, an independently obtained numerical solution, which utilizes a variable-order variable-step Adams method, provides possibly the best reference with which to assess the approximate model solutions. The numerical strategy for systematically locating the symmetric localized solutions is outlined in Champneys and Spence (1993).

Various criteria may be used to gauge the quality of an approximate solution obtained via Galerkin's method. While it may be argued that no single technique can adequately describe the error between the exact and approximate solutions, a number of different views are presented below which together confirm that the method outlined here has been used successfully to approximate localized solutions of eqn (7).

Figure 3 illustrates the close comparison achieved between the approximate deflected shapes, resulting from each order of the Galerkin analysis, and the independent numerical solution. As expected, the first order approximation provides the worst agreement, with successive improvement following the introduction of further modes. The fourth mode solution is indistinguishable, to the naked eye, from the numerical solution for all load values presented. The existence of a nontrivial buckled shape at $P = 0.0$ does not indicate an initial deflection of the geometry. Rather, it represents an equilibrium point on the post-buckling curve, albeit far from the initial buckling load, and possibly unlikely to be attained in a physical context.

The amplitude of the localized solution at the symmetric section ($x = 0$) is shown in Fig. 4. For a single mode approximation the amplitude is best predicted close to $P = P^c$ with worsening approximation as $P \to 0$. A significant improvement is obtained by the introduction of a second mode, and with four modes the approximation is once again obscured by the numerical solution.

Instead of comparing the solutions at a specific point, an average measure of the error is achieved by integrating the square of the residual along the length of the strut

$$R_{av} = \int_{-\infty}^{\infty} R(x, \bar{w})^2 \, dx. \tag{33}$$

This removes the dependence of the error measure upon the spatial variable x, leaving it dependent purely on the assumed mode form. The better the approximate shape, the smaller the error, as demonstrated in Fig. 5. Perhaps inevitably, the agreement of the first and second-order approximations become progressively worse as the load decreases from P^c the point of expansion for the perturbation scheme, and consequently the source of the

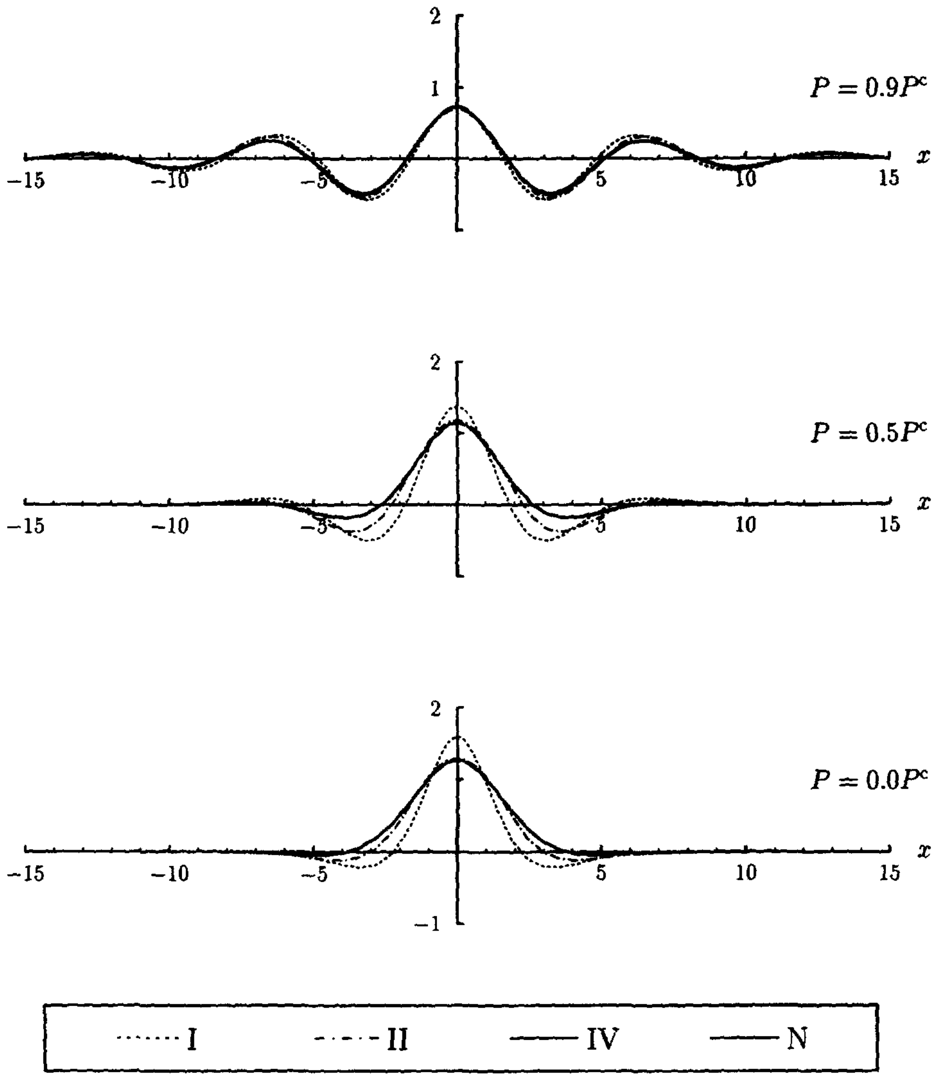

Fig. 3. Comparison of numerical and approximate buckle shapes for various loads: I—one mode; II—two modes; IV—four modes; N—numerical solution.

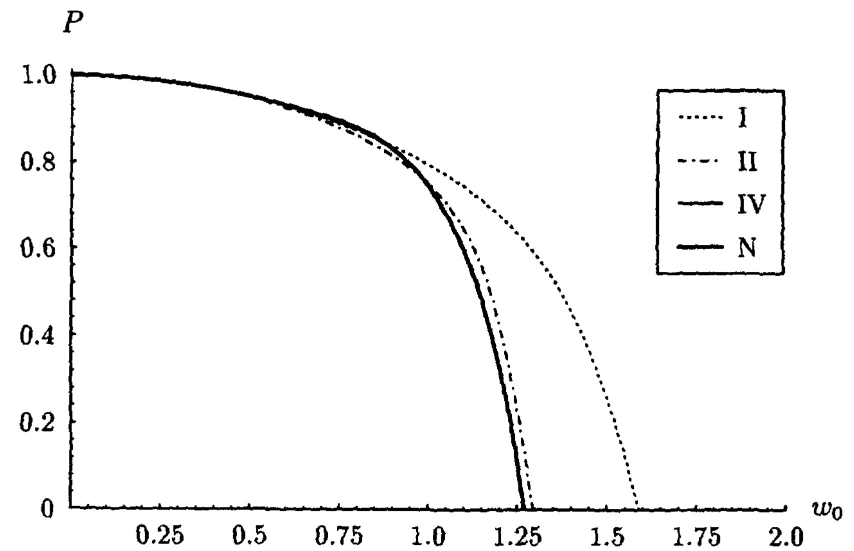

Fig. 4. Load vs amplitude (at $x = 0$) for single-hump localized solutions: I—one mode; II—two modes; IV—four modes; N—numerical solution.

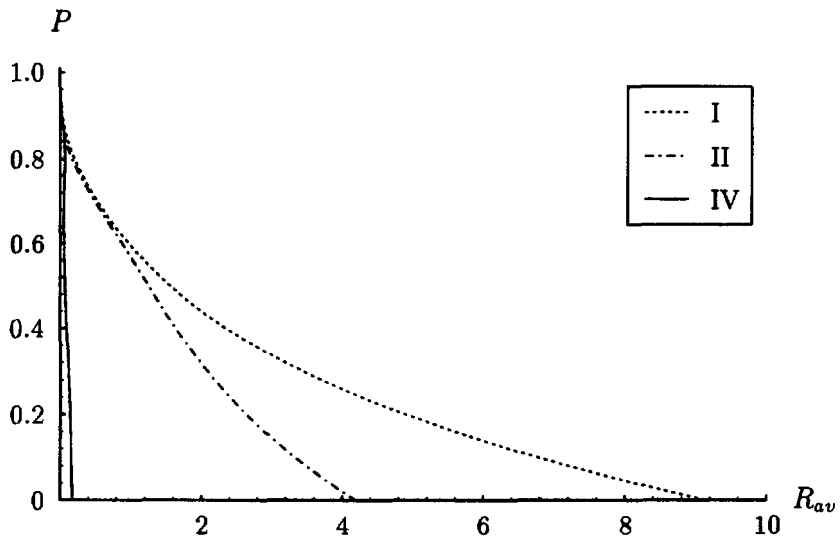

Fig. 5. Load vs average residual for approximate Galerkin solutions: I—one mode; II—two modes; IV—four modes.

mode shapes. Naturally, the exact solution, to which $\bar{w}$ tends as $n \to \infty$, has zero error. Although this method quantifies the error, it does so in an averaged sense, and is only useful when comparing one approximate form with another.

The first-order end-shortening is represented by

$$\Delta = \int_{-\infty}^{\infty} \frac{1}{2} w'^2 \, dx, \tag{34}$$

and, like the residual, may be considered as a measure of the average error between modal and numerical solutions. As seen in Fig. 6 the end-shortening is substantially over-estimated by a single mode approximation with a significant improvement following the inclusion of additional modes. The actual load vs end-shortening response is unusual with a turning point developing at approximately $P = 0.9P^c$. However, the appropriateness of the non-linear elastic foundation to model real behaviour is not of concern here—rather the ability of the approximate solution to mimic the behaviour of the accurate numerical solution. In

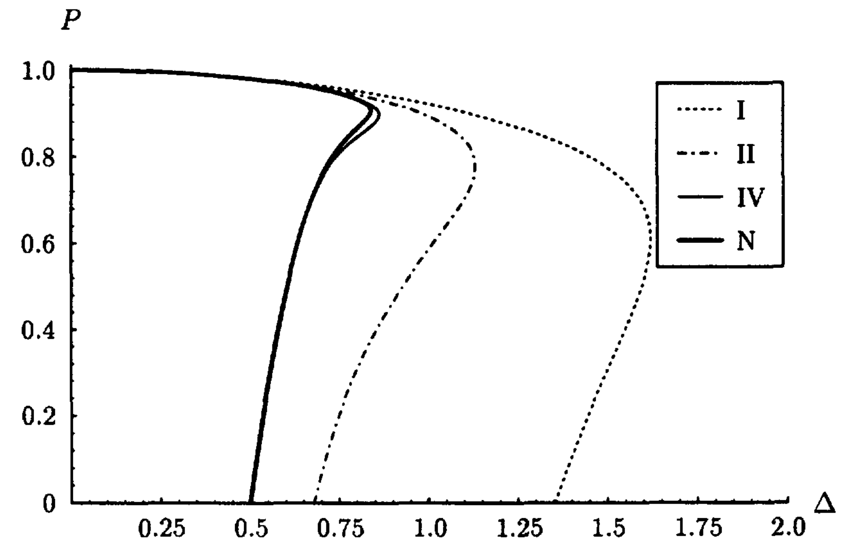

Fig. 6. Load vs end-shortening for single-hump localized solutions: I—one mode; II—two modes; IV—four modes; N—numerical solution.

any case, Fig. 6 represents the load vs end-shortening for a specific set of equilibrium solutions (the single-hump localized form) and is independent of the type of loading applied.

6. CONCLUDING REMARKS

An approximate solution technique has been developed to solve the problem of localized buckling of a strut on an elastic foundation. The method uses a continuous displacement function to represent each mode, with the actual shape being parameterized by the eigenvalues of the linearized differential equation, and the amplitude being found from a Galerkin procedure. It is apparent that all characteristics of the numerical solution are duplicated, to various degrees, by each approximate solution and that a very good agreement has been obtained with as few as four modes.

The application of a Galerkin procedure to problems of buckling in structural engineering is not a new exercise although the author is unaware of previous attempts to model the localized buckle pattern described in this paper. For the case of a purely elastic foundation, in which a total potential energy functional exists, other procedures based on an energy formulation, including the Raleigh–Ritz method, are equally valid. The similarities between the Galerkin method and the Rayleigh–Ritz method have long been known (Galerkin (1915)), and are currently being explored in the context of localized solutions (Wadee *et al.* (1996)).

Following the successful development of the Galerkin procedure described here it is hoped to extend the method to analyse the phenomenon of localized buckling of a strut on a visco-elastic foundation. In this case, a potential energy functional does not exist, making it necessary to base the method on the governing differential equation, thereby favouring the Galerkin approach.

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APPENDIX A

Second mode

The coefficients of the amplitudes in the coupled eqns (31) for a two mode approximation are:

$$
\begin{aligned}
k_1 &= \frac{1}{15\alpha}(15+7\alpha^4+30\alpha^2\beta^2+15\beta^4-10P(\alpha^2+3\beta^2)) \\
&\quad + \frac{\beta\pi}{15\alpha^2}(15+7\alpha^4+10\alpha^2\beta^2+3\beta^4-10P(\alpha^2+\beta^2))C_1, \\
k_2 &= \frac{4\beta}{15}(-7\alpha^2-5\beta^2+5P)+\frac{\beta^2\pi}{15\alpha^3}(15+7\alpha^4+10\alpha^2\beta^2+3\beta^4-10P(\alpha^2+\beta^2))C_1, \\
k_{111} &= -\frac{1}{2\alpha}-\frac{2\beta\pi}{3\alpha^4}(\alpha^2+\beta^2)C_1-\frac{\beta\pi}{3\alpha^4}(\alpha^2+4\beta^2)C_2, \\
k_{122} &= -\frac{1}{10\alpha}+\frac{\beta\pi}{5\alpha^6}(\alpha^4-16\beta^4)C_2, \\
k_{211} &= -\frac{\beta^2\pi}{2\alpha^5}(\alpha^2+\beta^2)C_1-\frac{\beta^2\pi}{\alpha^5}(\alpha^2+4\beta^2)C_2, \\
k_{222} &= \frac{\beta^2\pi}{90\alpha^7}(-7\alpha^4-5\alpha^2\beta^2+2\beta^4)C_1+\frac{\beta^2\pi}{45\alpha^7}(7\alpha^4+20\alpha^2\beta^2-32\beta^4)C_2,
\end{aligned}
\tag{A1}
$$

and

$$
\begin{aligned}
l_1 &= \frac{4\beta}{15}(-7\alpha^2-5\beta^2+5P)+\frac{\beta^2\pi}{15\alpha^3}(15+7\alpha^4+10\alpha^2\beta^2+3\beta^4-10P(\alpha^2+\beta^2))C_1, \\
l_2 &= \frac{1}{105\alpha}(35+155\alpha^4+294\alpha^2\beta^2+35\beta^4-P(98\alpha^2+70\beta^2)) \\
&\quad + \frac{\beta\pi}{105\alpha^4}(-35\alpha^2-155\alpha^6+70\beta^2-196\alpha^4\beta^2-35\alpha^2\beta^4+6\beta^6+P(98\alpha^4+70\alpha^2\beta^2-28\beta^4))C_1, \\
l_{111} &= -\frac{\beta^2\pi}{6\alpha^5}(\alpha^2+\beta^2)C_1-\frac{\beta^2\pi}{3\alpha^5}(\alpha^2+4\beta^2)C_2, \\
l_{122} &= \frac{\beta^2\pi}{30\alpha^7}(-7\alpha^4-5\alpha^2\beta^2+2\beta^4)C_1+\frac{\beta^2\pi}{15\alpha^7}(7\alpha^4+20\alpha^2\beta^2-32\beta^4)C_2, \\
l_{211} &= -\frac{1}{10\alpha}+\frac{\beta\pi}{5\alpha^6}(\alpha^4-16\beta^4)C_2, \\
l_{222} &= -\frac{3}{70\alpha}+\frac{2\beta\pi}{315\alpha^8}(9\alpha^6-7\alpha^4\beta^2-14\alpha^2\beta^4+2\beta^6)C_1+\frac{\beta\pi}{315\alpha^8}(-9\alpha^6+28\alpha^4\beta^2+224\alpha^2\beta^4-128\beta^6)C_2,
\end{aligned}
\tag{36}
$$

where C_1 and C_2 are given in (28).